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Rigid limit in $\mathcal{N} = 2$ supergravity and weak-gravity conjecture

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Abstract: We analyze the coupled $\mathcal{N} = 2$ supergravity and Yang-Mills system using holomorphy, near the rigid limit where the former decouples from the latter. We find that there appears generically a new mass scale around $gM_{\rm pl}$ where g is the gauge coupling constant and $M_{\rm pl}$ is the Planck scale. This is in accord with the weak-gravity conjecture proposed recently. We also study the scale dependence of the gauge theory prepotential from its embedding into supergravity.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, String Duality, Renormalization Group.

Contents

1. Introduction

Quantization of general relativity has been one of the most serious challenges for theoretical physics for a long time. Its coupling constant is dimensionful, which makes the theory apparently non-renormalizable. Thus, we need to complete the theory in the ultraviolet (UV) to make it into a consistent quantum theory. The prime candidate for quantized gravity is the superstring theory, and the progress we made during the last decade makes us confident that there exist many consistent four-dimensional theories with a high degree of supersymmetry containing quantized graviton in their spectrum. These low energy field theories coupled to gravity have a consistent UV completion and are obtained via compactification of superstring theory on suitable internal manifolds.

When we come to theories with a smaller number of supersymmetries the situation becomes somewhat delicate. Recent developments suggest that there exists an enormous number of $\mathcal{N} = 1$ supersymmetric four-dimensional models with negative cosmological constant (for a review, see e.g. [1]). This *landscape* of superstring vacua, if taken at face value, predicts a disturbingly huge number, 10^{200} or larger, of solutions with varying gauge groups and matter contents. Then it is natural to ask which theory is realized as a low-energy effective description of a consistent theory with quantized gravity [2]. Several criteria have been already proposed in [3, 4] which characterize models in the *swampland* which cannot be UV completed to a consistent theory of quantum gravity.

The criterion we will focus in this article is *the weak-gravity conjecture* proposed in [3]; one way to state the conjecture is that if a consistent theory coupled to gravity with the Planck scale $M_{\rm pl}$ contains a gauge field with the coupling constant g, then there should

JHEP08(2007)068

necessarily be a new physics around the mass scale $gM_{\rm pl}$. We refer the reader to the original article for the arguments which led to this proposal [3]. Our objective in this article is to show how this conjecture will generically hold within the framework of $\mathcal{N}=2$ supersymmetric Yang-Mills coupled to $\mathcal{N}=2$ supergravity.

The system of $\mathcal{N}=2$ supersymmetry is well suited to the analysis of the effects of quantum gravity on the gauge theory. One advantage is that the dynamics of $\mathcal{N}=2$ supersymmetric Yang-Mills theories has been studied in great detail since the pioneering work of [5]. Another advantage is that the limit where the $\mathcal{N}=2$ Yang-Mills theory decouples from the $\mathcal{N}=2$ supergravity is fairly well understood in the context of the string compactification on Calabi-Yau (CY) manifold with a fiber of ADE singularities. This limit is known as the *rigid limit* or *decoupling limit* since supersymmetry becomes rigid and gravity decouples from the gauge theory in the limit. It is also called the *geometric-engineering limit* [6–8], since non-Abelian gauge symmetry is generated by ADE singularities.

In this paper we consider a type II string theory on CY manifolds which possess $K3$ fibration over $\mathbb{CP}^1$ and thus has a dual heterotic string description. At the geometric engineering limit $\epsilon \to 0$ when the $K3$ surface develops ADE singularity, such a CY manifold acquires periods which behave as a power and logarithm of ϵ. We shall show that the ratio of these periods leads to the hierarchy of gauge and gravity mass scales which has exactly the form of the weak gravity conjecture. Since the geometric engineering limit is the only way to generate non-Abelian gauge symmetry in type II theory, the weak gravity conjecture seems to hold generically in $\mathcal{N}=2$ gauge theory coupled to $\mathcal{N}=2$ supergravity. Actually as is well-known, $M_{\rm het}{=}gM_{\rm pl}$ is the mass scale of heterotic string theory and thus the weak gravity conjecture seems to fit very nicely with the type II-heterotic duality.

In our analysis the holomorphy and the special geometry of $\mathcal{N}=2$ theories play the basic role. Holomorphic functions are determined by their behavior at the singularities, in particular by the monodromy properties around the singular locus.

The organization of the paper is as follows: In section 2 we discuss an example of a type II string theory compactified on a CY manifold with a $K3$ fibration. We shall show how a hierarchy of mass scales is generated in the rigid limit $\epsilon \to 0$ which fits exactly to the weak gravity conjecture. We also point out that the presence of a logarithmic period $\log \epsilon$ predicts a kinetic term for a field S

$$\frac{\partial_\mu S \partial_\mu S}{(\mathrm{Im}\, S)^2}. \tag{1.1}$$

S corresponds to the gauge coupling constant $S=\theta/(2\pi)+4\pi i/g^2$ and maps to the heterotic dilaton under the type II/heterotic duality. In section 3 we will discuss generalization of the weak gravity hypothesis. We discuss in section 4 the mechanism of how the logarithmic periods necessarily appear in a CY manifold with a $K3$ fibration. In section 5 we use the embedding of gauge theory into supergravity and derive the *renormalization group* formula for the dependence of the prepotential $F_{\rm gauge}$ on the dynamical scale Λ [9–11]. We derive for any gauge theory of ADE group, a relation

$$\frac{\partial F_{\rm gauge}}{\partial \log \Lambda} = \frac{h}{\pi i} u_2. \tag{1.2}$$

Here $u_2 = \langle \mathrm{tr}\, \phi^2 \rangle$ and ϕ denotes the adjoint scalar in the vector multiplet. h is the Coxeter number of the group. We conclude this note with some discussions in section 6.

2. An example

2.1 A Calabi-Yau and the rigid limit

Let us start with an example from the string theory. As is well-known, in the type IIA superstring theory, an $\mathcal{N} = 2$ supergravity system in four dimensions can be obtained by compactification on a CY manifold M. It is also known that the $\mathrm{SU}(n)$ $\mathcal{N} = 2$ gauge symmetry arises if M has a sphere of A_{n-1} type singularities. In the simplest case of A_1 singularity such a CY manifold has at least two Kähler parameters: one for the size of the sphere of the singularities, and the other for the size of resolution of singularities. One explicit example is given by a CY manifold X_8 which is a degree 8 hypersurface in the weighted projective space $\mathbb{WCP}^4_{1,1,2,2,2}$ with Hodge numbers $h_{11} = 2$, $h_{21} = 86$.

Our analysis is facilitated by going to the mirror type IIB theory where world-sheet instanton corrections in IIA theory are summed up by mirror transformation. Mirror pair of X_8 and X_8^* has been extensively studied in the literature (e.g. [12–14]). We first briefly review their properties. Defining equation of the mirror X_8^* is given by

$$X_8^* : \; W = \frac{B}{8}x_1^8 + \frac{B}{8}x_2^8 + \frac{1}{4}x_3^4 + \frac{1}{4}x_4^4 + \frac{1}{4}x_5^4 - \psi_0 x_1x_2x_3x_4x_5 - \frac{1}{4}\psi_2(x_1x_2)^4 = 0 \qquad (2.1)$$

in an orbifold of $\mathbb{WCP}^4_{1,1,2,2,2}$. $[B : \psi_0 : \psi_2]$ parametrizes the complex structure moduli of X_8^*. We first note that this hypersurface has a structure of a $K3$ fibration over $\mathbb{CP}^1$: by a change of variables $x_0 = x_1x_2$, $\zeta = x_1/x_2$, W is rewritten as

$$W = \frac{B'}{4}x_0^4 + \frac{1}{4}x_3^4 + \frac{1}{4}x_4^4 + \frac{1}{4}x_5^4 - \psi_0 x_0x_3x_4x_5 = 0, \qquad (2.2)$$

$$B' = \frac{B}{2}\left(\zeta + \frac{1}{\zeta}\right) - \psi_2. \qquad (2.3)$$

ζ parametrizes the base of the $K3$ fibration. $K3$ surface (2.2) (with fixed ζ) has singularities at

$$B' = 0; \qquad \text{large complex structure limit,} \qquad (2.4)$$

$$B' = \psi_0^4; \qquad \text{conifold singularity.} \qquad (2.5)$$

These are located by imposing equations $W = 0$, $\partial W/\partial x_i = 0$, $i = 0, 3, 4, 5$ simultaneously. If we solve (2.4), (2.5) for ζ, we find

$$B' = 0 \Longrightarrow \zeta = e_0^{\pm}, \;\text{ where }\; e_0^{\pm} = \frac{\psi_2}{B} \pm \sqrt{\left(\frac{\psi_2}{B}\right)^2 - 1}, \qquad (2.6)$$

$$B' = \psi_0^4 \Longrightarrow \zeta = e_1^{\pm}, \;\text{ where }\; e_1^{\pm} = \frac{(\psi_2 + \psi_0^4)}{B} \pm \sqrt{\left(\frac{\psi_2 + \psi_0^4}{B}\right)^2 - 1}. \qquad (2.7)$$

JHEP08(2007)068

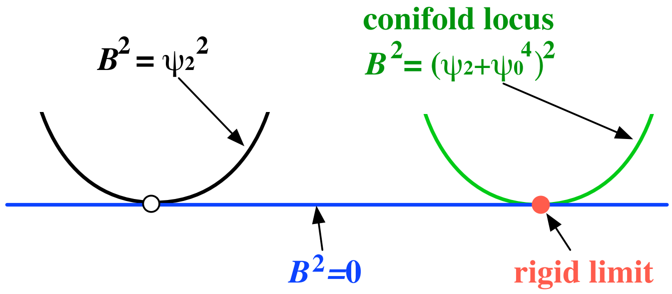

Figure 1: Discriminant loci of the moduli of the CY X_8^*, before the blowup.

Singularities of the total space X_8^* are located by further imposing $\partial B'/\partial\zeta = 0$

$$\frac{\partial B'}{\partial \zeta} \Longrightarrow \quad B = 0 \quad \text{or} \quad \zeta = \pm 1. \tag{2.8}$$

Substituting $\zeta = \pm 1$ into (2.4), (2.5) we find singular loci in the moduli space of X_8^*

$$B = \pm\psi_2, \quad B = \pm(\psi_2 + \psi_0^4). \tag{2.9}$$

These coincide with the locations where $e_0^\pm$, $e_1^\pm$ become degenerate.

Thus the discriminant of the mirror CY manifold is given by

$$\Delta = B^2(B^2 - \psi_2^2)(B^2 - (\psi_2 + \psi_0^4)^2). \tag{2.10}$$

Three components of the discriminant loci are depicted in figure 1. The first and the second factor intersect tangentially at the large complex structure point, and the third factor is the conifold locus. The conifold locus and the locus $B^2 = 0$ also meet tangentially at the rigid limit,[1] so that the moduli space needs to be blown up at these points.

We now concentrate on the region near the rigid limit. The blowing up introduces an exceptional curve which is a $\mathbb{CP}^1$ parametrized by $[\Lambda^2 : u]$ via the relation

$$\epsilon\Lambda^2 = B, \qquad \epsilon u = \psi_2 + \psi_0^4. \tag{2.11}$$

The exceptional curve is at $\epsilon = 0$. The discriminant loci after the blowup are shown in figure 2.

The defining polynomial W in the limit $\epsilon \to 0$ is given by

$$W = \frac{\epsilon}{2}\left[\frac{1}{2}\left(w + \frac{\Lambda^4}{w}\right) + x^2 + y^2 + z^2 - u\right] + O(\epsilon^2). \tag{2.12}$$

after a suitable redefinition of the coordinates. This is a fibration of A_1 singularity over $\mathbb{CP}^1$ parametrized by w. It is in fact the Seiberg-Witten geometry of the $\mathcal{N} = 2$ supersymmetric pure SU(2) Yang-Mills theory with the modulus $u = \langle \mathrm{tr}\,\phi^2 \rangle$ and the dynamical mass scale Λ. Thus, the exceptional curve we have introduced is identified as the u-plane of SU(2)

[1]The parameter sets (B, ψ_0, ψ_2) and $(-B, \psi_0, \psi_2)$ describe the same complex structure, and so the natural coordinate of the moduli is B^2 rather than B.

JHEP08(2007)068

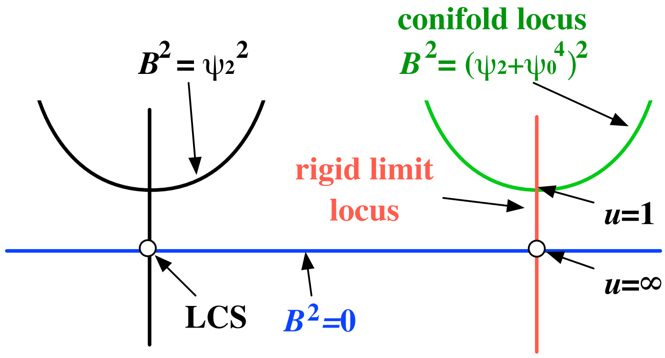

Figure 2: Discriminant loci of the moduli of the CY X_8^* after the blowup. LCS stands for the Large Complex Structure point.

gauge theory: the u-plane is naturally compactified at $u = \infty$ into a sphere. We call this sphere the rigid limit locus.

Note that before taking the rigid limit $\epsilon \to 0$, the theory contains $h_{11} + 1 = 3$ gauge fields: they are the graviphoton, the gauge partner of the scalar field S and the U(1) (Cartan-subalgebra) part of SU(2) gauge field. Here S denotes the scalar field which corresponds to the gauge coupling constant in field theory,

$$S = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{2.13}$$

We recall that when CY manifold M possesses a $K3$ fibration on $\mathbb{CP}^1$, there exist a duality between type IIA on M and heterotic theory on $K3 \times T^2$ [15]. The field S corresponds to the size of the base $\mathbb{CP}^1$ of $K3$ fibration in type IIA theory and becomes the heterotic dilaton under this duality. In the decoupling limit $\epsilon \to 0$, two of the gauge fields, the graviphoton and the partner of S, disappear and we are only left with the (Cartan part of) SU(2) gauge field.

2.2 Behavior of the Kähler potential

Let us next quickly recall the structure of vector multiplet scalars in the $\mathcal{N} = 2$ theories. First, in the case of field theories of rigid $\mathcal{N} = 2$ supersymmetry with the gauge group $\mathrm{U}(1)^n$, there exist n complex scalar fields ϕ^i, $(i = 1, \ldots, n)$. Their Kähler potential is given by

$$K = \mathrm{Im} \sum_i (a_i^D)^* a^i \tag{2.14}$$

where a^i and a_i^D are holomorphic functions of the VEV's of ϕ^i. a^i and a_i^D are called the special coordinates or the periods of the theory. Dual periods are related to each other as

$$a_i^D = \frac{\partial F_{\mathrm{gauge}}}{\partial a^i}, \quad i = 1, \cdots, n \tag{2.15}$$

where F_{gauge} denotes the prepotential.

Secondly, in the case of $\mathcal{N}=2$ supergravity with N vector multiplets, there exist $2(N+1)$ periods X^a, F_a, $a=1,\cdots,N+1$. The Kähler potential is given by

$$e^{-K}=\mathrm{Im}\sum_a F_a^* X^a. \tag{2.16}$$

The periods X^a, F_a are holomorphic functions of scalars Φ^i, $(i=1,\ldots,N)$. Under the Kähler transformation $K\to K-f-f^*$ periods are transformed as $X^a\to e^f X^a$, $F_a\to e^f F_a$. The mass squared of a BPS-saturated soliton with charges (q_a, m^a) is then given by

$$m^2=e^K|\sum_a(q_a X^a+m^a F_a)|^2, \tag{2.17}$$

which is invariant under the Kähler transformation. An important property of the supergravity periods is the transversality condition:

$$\sum_a X^a\frac{\partial F_a}{\partial\Phi^i}-\sum_a\frac{\partial X^a}{\partial\Phi^i}F_a=0, \tag{2.18}$$

which guarantees the existence of the prepotential. Prepotential of $\mathcal{N}=2$ supergravity is a homogeneous function of degree 2 in X_a.

In the case of CY compactification of type IIB string theory, the periods are given by

$$X^a=\int_{A^a}\Omega,\qquad F_a=\int_{B_a}\Omega \tag{2.19}$$

where Ω is the $(3,0)$-form of the CY and A^a, B_a are the canonical basis of $H_3(M^*,\mathbb{Z})$ of CY manifold. In this case the condition (2.18) comes from the Griffiths transversality $\int\Omega\wedge\partial_{\Phi^i}\Omega=0$.

Now let us go back to the example of the previous section, type IIB string theory compactified on X_8. In the field theory limit we have only one gauge field ($n=1$) and two periods a and a^D of SU(2) Seiberg-Witten theory. At the level of supergravity there exist three gauge fields (two vector multiplets, $N=2$) and six periods X^a, F_a, $a=1,2,3$. Behavior of these periods near the decoupling limit and in particular their monodromy properties around rigid limit locus have been discussed in great detail in [14].

It turns out that two of the periods, say X^1 and F_1, are converted to the gauge theory periods in the rigid limit. They behave as

$$X^1=\epsilon^{1/2}a+O(\epsilon),\qquad F_1=\epsilon^{1/2}a^D+O(\epsilon). \tag{2.20}$$

Remaining four periods behave as

$$X^2,\, X^3=1+O(\epsilon^{1/2}),\qquad F_2,\, F_3=\frac{1}{2\pi i}\log\epsilon+O(1). \tag{2.21}$$

The origin of logarithmic behaviors in F_2, F_3 will be discussed in section 4: they come from the geometry of $K3$ fibration of the CY manifold.

Then using (2.16) we find that e^K behaves as $\log 1/|\epsilon|$. Therefore the supergravity Kähler potential is expanded as

$$K=\log(\log 1/|\epsilon|)+\frac{|\epsilon|}{\log 1/|\epsilon|}\,\mathrm{Im}(a^D)^* a+\cdots. \tag{2.22}$$

as $\epsilon \to 0$. Note that $\mathrm{Im}\,\left(a^D\right)^* a$ is the Kähler potential of the field theory (2.14). Thus we can clearly see that SU(2) super Yang-Mills theory decouples from gravity.

The factor $|\epsilon|$ in front of the Kähler potential of the field theory determines the hierarchy between the Planck scale and the scale of the gauge theory: it is basically in accord with the expectation [8] with $|\epsilon|^{1/2}$ being identified with the dynamical mass scale Λ_{gauge} of the gauge theory. The existence of an extra factor of $\log 1/|\epsilon|$ in the denominator was first recognized by the authors of [14]. We will see in the following that this factor implies the weak-gravity conjecture in the present context.

Let us now consider the weak coupling region of gauge theory for the sake of simplicity. There the periods a and a^D behave as

$$a \approx \sqrt{2u}, \qquad a^D \approx \frac{i}{\pi}\sqrt{2u}\log u. \tag{2.23}$$

Using the relation of periods to the low-energy gauge coupling constant τ:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2(m_W)} = \frac{\partial a^D}{\partial a}, \tag{2.24}$$

we find

$$e^{-2\pi^2/g^2(m_W)} = u^{-1/2}. \tag{2.25}$$

The coupling constant g in the above equation is to be evaluated at the scale of the mass m_W of the massive gauge boson where the coupling stops running. m_W is, in turn, given by the formula (2.17)

$$m_W^2 = e^K |X^1|^2 = \frac{|\epsilon|}{\log 1/|\epsilon|} u. \tag{2.26}$$

From (2.25) and (2.26), we find the dynamical scale of the gauge theory

$$\Lambda_{\mathrm{gauge}} = m_W e^{-2\pi^2/g^2(m_W)} = \frac{|\epsilon|^{1/2}}{(\log 1/|\epsilon|)^{1/2}} M_{\mathrm{pl}} \tag{2.27}$$

where we reinstated the Planck scale to recover the correct mass dimension.

Let us next introduce a chiral superfield $S = \theta/2\pi + 4\pi i/g^2$ via the relation

$$S = \frac{1}{\pi i}\log \epsilon. \tag{2.28}$$

Then, the monodromy around $\epsilon = 0$ is generated by the shift $S \to S+2$. $\mathrm{Im}\, S$, which is the partner of the dynamical theta angle, is the natural bare gauge coupling constant in the supergravity. Furthermore, S coincides with the heterotic dilaton which we have discussed at the end of section 2.1. There will be subleading corrections to (2.28) if one goes outside the region of weak coupling or small ϵ. Another notable fact is that, because of the Kähler potential (2.22), the field S in fact has the standard kinetic term for the dilaton,

$$g_{SS^*}\partial_\mu S \partial_\mu S^* = \frac{\partial_\mu S \partial_\mu S^*}{(\mathrm{Im}\, S)^2}. \tag{2.29}$$

Using the field $S = \theta/2\pi + 4\pi i/g^2$, the relation (2.27) now becomes

$$\Lambda_{\mathrm{gauge}} = e^{-2\pi^2/g^2} \cdot g M_{\mathrm{pl}}. \tag{2.30}$$

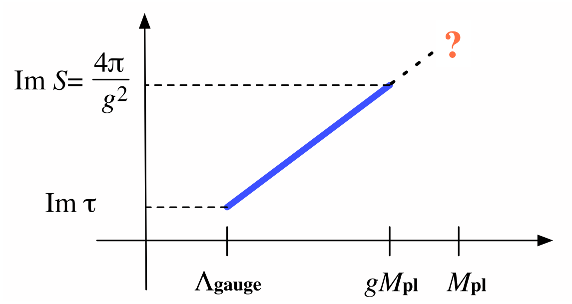

Figure 3: Running of the coupling in the gauge theory coupled to supergravity.

There exists an extra factor of g in front of $M_{\rm pl}$ in the above equation, which means that the ultraviolet gauge coupling g is defined not at the Planck scale $M_{\rm pl}$ but at a lower energy scale $gM_{\rm pl}$. The running of the gauge coupling from the value at low energy $\operatorname{Im}\tau$ to the one at high energy $\operatorname{Im} S$ is schematically depicted in figure 3. The existence of the new scale $gM_{\rm pl}$ is what the weak gravity conjecture has predicted. Thus the analysis of the $\mathcal{N}=2$ SU(2) gauge theory coupled to supergravity supports the weak gravity conjecture.

3. Generalization

Let us consider what happens in the generic $\mathcal{N}=2$ gauge theory coupled to $\mathcal{N}=2$ supergravity. Suppose the gauge theory has n vector multiplets. In the coupled gauge-gravity system the gauge coupling constant is promoted to a scalar field S. Thus there is at least one extra vector multiplet in the locally supersymmetric theory. Let us consider the minimal situation; i.e. the total number of the U(1) vector multiplets being equal to $n+1$. Then, altogether there are $n+2$ gauge fields including the graviphoton and there will be $2n+4$ supergravity periods. Therefore, by coupling the gauge theory to the $\mathcal{N}=2$ gravity we should have at least four extra periods.

We assume that there is a locus E in the suitably blown-up moduli space given by the local parameter $\epsilon=0$ around which some Ω_i, $(i=1,\ldots,2n)$ of the periods Ω_I, $(I=1,\ldots,2(n+2))$ become parametrically small, $\Omega_i \propto O(\epsilon^{1/h})$ for some power h. This statement itself is not invariant under Kähler transformation, so we also demand that there will be periods which stay constant near E.

The monodromy around E may also be logarithmic: thus there might be periods behaving as $\propto (\log\epsilon)^k$. Let p be the largest power k of such periods. There is a mathematical theorem[2] which then states the Kähler potential behaves as

$$e^{-K}=\operatorname{Im}\sum_a F_a^* X^a \propto (\log|\epsilon|)^p. \tag{3.1}$$

Let us define the chiral field S by

$$S=\frac{1}{\pi i}\log\epsilon=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2} \tag{3.2}$$

[2]see e.g. appendix A of [16] and references therein.

JHEP08(2007)068

as before. Repeating the argument presented in the last section, we readily obtain a relation

$$\Lambda_{\rm gauge} \sim e^{-4\pi^2/(hg^2)} \cdot g^p M_{\rm pl}. \tag{3.3}$$

We will see in the next section that h equals the quadratic Casimir of the gauge group in the case of pure $\mathcal{N}=2$ gauge theory.

Furthermore, the kinetic term of S is given by

$$\frac{\partial_\mu S \partial_\mu S^*}{(\mathrm{Im}\, S)^2} \qquad \text{or} \qquad \partial_\mu S \partial_\mu S^* \tag{3.4}$$

depending on $p \neq 0$ or $p=0$, respectively. Thus, the weak gravity conjecture in $\mathcal{N}=2$ supergravity coupled to super Yang-Mills follows from the existence of a logarithmic period $\sim (\log\epsilon)^p$, $p \geq 1$. Furthermore, the appearance of such logarithmic periods is related to the field $S = \log\epsilon$ corresponding to the dilaton in heterotic theory.

4. Explicit description of logarithmic periods

For a CY which is a $K3$ fibration over $\mathbb{CP}^1$, 3-cycles can be constructed explicitly. We follow the approach of [14] and appendix in [17] . Consider a CY with a defining equation

$$w + \frac{\mu^2}{w} + W_{K3}(x,y,z;t_\ell) = 0 \tag{4.1}$$

where t_ℓ denote the moduli of the $K3$. The holomorphic 3-form is given by

$$\Omega = \frac{dw}{w} \wedge \Omega_{K3}, \qquad \Omega_{K3} = \frac{dx \wedge dy}{\partial_z W_{K3}}. \tag{4.2}$$

3-cycles of CY are made of the product of a 1-cycle of the $\mathbb{CP}^1$ base and a 2-cycle of $K3$.

2-cycles of $K3$ to be used here are those which are not holomorphically embedded into $K3$, since holomorphic cycles have the representative which are of the $(1,1)$-form so that their integrals with the (2,0)-form Ω_{K3} must vanish. Holomorphic cycles of $K3$ form the Picard lattice $\Lambda^{\rm Pic}$

$$\Lambda^{\rm Pic} = H^{1,1}(K3) \cap H^2(K3,\mathbb{Z}) \tag{4.3}$$

and its dimension is called the Picard number $\rho(K3)$. Cycles which are not holomorphically embedded are called transcendental and the lattice Λ of the 2nd homology of $K3$ has an orthogonal decomposition into Picard and transcendental lattices

$$\Lambda = \Lambda^{\rm Pic} \oplus \Lambda^{\rm tr}. \tag{4.4}$$

It is well-known that the lattice Λ has a signature of $(3,19)$. In the case of projective $K3$, the Kähler form becomes algebraic and the Picard lattice has a signature $(1,\rho(K3)-1)$. Then the signature of $\Lambda^{\rm tr}$ becomes $(2, 20-\rho(K3))$.

In the case of the quartic $K3$ surfaces which featured in our example X_8, the Picard number is $\rho(K3) = 19$ and thus there are three transcendental cycles with signature $(2,1)$. The 2-cycle with a negative signature, i.e. a negative self-intersection number is the vanishing cycle of A_1 singularity. Two 2-cycles of the positive signature generate periods which

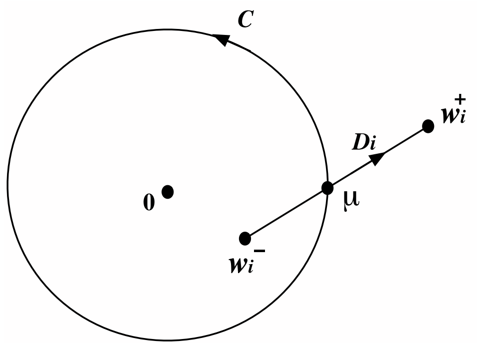

Figure 4: Cuts in the base $\mathbb{CP}^1$.

	defining eq.	h	d_x	d_y	d_z	deg. of Casimirs
A_{n-1}	$0 = x^2 + y^2 + z^n$	n	$n/2$	$n/2$	1	$2, 3, \ldots, n$
D_{n+1}	$0 = x^2 + y^2 z + z^n$	$2n$	n	$n-1$	2	$2, 4, \ldots, 2n$
E_6	$0 = x^2 + y^3 + z^4$	12	6	4	3	$2, 5, 6, 8, 9, 12$
E_7	$0 = x^2 + y^3 + yz^3$	18	9	6	4	$2, 6, 8, 10, 12, 14, 18$
E_8	$0 = x^2 + y^3 + z^5$	30	15	10	6	$2, 8, 12, 14, 18, 20, 24, 30$

Table 1: Data of ADE singularities.

have logarithmic behavior in ϵ as we see below. Ref. [14] discusses another example of CY manifold X_{24} which also possesses a $K3$ fibration and produces the SU(3) gauge theory in the decoupling limit. In this case there exist four transcendental cycles with a signature $(2, 2)$. Two 2-cycles with the negative signature describe the vanishing cycles of A_2 singularity. In the case of general A_r singularity there will be $2 + r$ transcendental cycles with the signature $(2, r)$. As we shall see below, two transcendental cycles of $K3$ with the positive signature will generate logarithmic cycles of CY manifold.

The CY (4.1) can be thought of as a one-parameter family of $K3$, whose moduli depend on w. Suppose a transcendental two-cycle S_i degenerates at $w + \mu^2/w = k_i$. For a small μ, this happens at $w_i^+ \sim k_i$ and $w_i^- \sim \mu^2/k_i$, see figure 4. Let C be the circle around the origin $|w| = |\mu|$, and D_i denote the path connecting $w_i^\pm$. Then $C \times S_i$ and $D_i \times S_i$ are closed 3-cycles of CY manifold.

In general, Yang-Mills gauge theories are geometrically engineered by fine-tuning the parameters $\{t_\ell\}$ of $K3$ so that the $K3$ develops ADE singularities: see [7] for SU(n), [18] for SO(n) and [19, 20] for E_n groups. Suppose we have a singularity of type G with rank $G = r$ around $x = y = z = 0$. The moduli $\{t_\ell\}$ of $K3$ are decomposed into two sets of parameters

$$\{u_2, \cdots, u_h\}, \quad \{v_1, v_2, \cdots\}, \tag{4.5}$$

where u_i corresponds to the degree i Casimir invariant of the group G. u_i are tuned to vanish as $\epsilon^{i/h}$ in the geometric engineering limit and we rescale them as $\epsilon^{i/h} \cdot u_i$. Here h is the dual Coxeter number of G. v_j are the moduli which remain finite in the engineering

JHEP08(2007)068

limit. We also introduce the rescaled coordinates as

$$w=\epsilon\tilde{w},\quad x=\epsilon^{d_x/h}\tilde{x},\quad y=\epsilon^{d_y/h}\tilde{y},\quad z=\epsilon^{d_z/h}\tilde{z}. \tag{4.6}$$

$d_{x,y,z}$ are the degrees of x,y,z (see table 1). We also set $\mu=\epsilon\Lambda^h$. Then the defining equation (4.1) of the CY becomes

$$\epsilon\left(\tilde{w}+\frac{\Lambda^{2h}}{\tilde{w}}+W_{ADE}(\tilde{x},\tilde{y},\tilde{z};u_i)+O(\epsilon^{1/h})\right)=0. \tag{4.7}$$

The holomorphic 3-form is given by

$$\Omega=\frac{dw}{w}\wedge\frac{dx\wedge dy}{\partial_z W_{K3}}=\epsilon^{(d_x+d_y+d_z)/h-1}\frac{d\tilde{w}}{\tilde{w}}\wedge\frac{d\tilde{x}\wedge d\tilde{y}}{\partial_{\tilde{z}}W_{ADE}}=\epsilon^{1/h}\frac{d\tilde{w}}{\tilde{w}}\wedge\Omega_{ADE} \tag{4.8}$$

where we used the fact $d_x+d_y+d_z=h+1$.

There are r independent two-cycles S_i of $K3$ which vanish simultaneously in the engineering limit. These give rise to $2r$ 3-cycles $\bar{A}_i=C\times S_i$ and $\bar{B}_i=D_i\times S_i$, $i=1,\cdots,r$ of the CY as explained above. We can take their linear combinations, A^i and B_i, so that they have the canonical intersection form, $(A^i,A^j)=(B_i,B_j)=0$, $(A^i,B_j)=\delta^i_j$. Then

$$a^i=\int_{A^i}\frac{d\tilde{w}}{\tilde{w}}\wedge\Omega_{ADE},\qquad a_i^D=\int_{B_i}\frac{d\tilde{w}}{\tilde{w}}\wedge\Omega_{ADE}, \tag{4.9}$$

are identified with the special coordinates of Seiberg-Witten theory. Corresponding supergravity periods behave as

$$X^i=\int_{A^i}\Omega=\epsilon^{1/h}a^i+O(\epsilon^{2/h}),\qquad F_i=\int_{B_i}\Omega=\epsilon^{1/h}a_i^D+O(\epsilon^{2/h}). \tag{4.10}$$

$K3$ surface has two extra 2-cycles which have a positive signature, as we have noted above. We call them T_a, $a=1,2$ and arrange them so that they do not intersect S_i and stay at finite values of x, y and z. Now the defining equation of CY near the 3-cycles T_a is given by

$$w+\frac{\epsilon^2\Lambda^{2h}}{w}+W_{K3}(x,y,z;0,v_j)=0 \tag{4.11}$$

Thus from the cycle $U_a=C\times T_a$ we obtain the period

$$\Omega_{U_a}=\int_{U_a}\Omega=\oint_C\frac{dw}{w}\int_{T_a}\Omega_{K3}=2\pi i\,c_a\approx O(1)\quad\text{where}\quad c_a=\int_{T_a}\Omega_{K3}(u_i=0;v_j). \tag{4.12}$$

In the case of the cycles $V_a=D_a\times T_a$, the end points of the w integration become

$$w_a^-\sim\frac{\epsilon^2\Lambda^{2h}}{k_a},\quad w_a^+\sim k_a \tag{4.13}$$

where $k_a=w+\epsilon^2\Lambda^{2h}/w$ is the value at which the 2-cycle T_a degenerates. Then we find the logarithmic behavior

$$\Omega_{V_a}=\int_{V_a}\Omega=\int_{w_a^-}^{w_a^+}\frac{dw}{w}\int_{T_a}\Omega_{K3}\approx-2c_a\log\epsilon. \tag{4.14}$$

The analysis of the monodromy under the phase rotation of ϵ suggests

$$\Omega_{V_a} \approx -\frac{1}{\pi i} \log \epsilon \cdot \Omega_{U_a} + O(\epsilon^{1/h}), \tag{4.15}$$

although the precise form of this expression will depend on the intersection form of T_a. Thus we have established the existence of periods behaving logarithmically near the engineering limit.

5. Renormalization group equation

As an application of the above analysis, we shall derive the relation

$$\frac{\partial F_{\text{gauge}}}{\partial \log \Lambda} = \frac{h}{\pi i} u_2 \tag{5.1}$$

for pure $\mathcal{N} = 2$ Yang-Mills theory with gauge groups $G = A, D, E$ from its embedding into supergravity. Here $u_2 = \langle \operatorname{tr} \phi^2 \rangle$ is the second order Casimir and is a monodromy-invariant coordinate of the moduli space. The relation describes the scaling violation of the prepotential of gauge theory and is called the renormalization group equation.

Before we start describing our derivation, let us recall how the equation (5.1) was obtained from the point of view of the gauge theory. Originally it was derived for SU(2) in [9] using the Picard-Fuchs equation for Seiberg-Witten curve, and later it was generalized to the classical gauge groups in [10, 11] using the property of the hyperelliptic curve describing the dynamics of the theory. For the E-type gauge groups the relation has not been given from the SW curve because of its complexity; thus our method gives the first verification of the relation for the E-type gauge groups.

The relation has been used in the analysis of the geometrical engineering limit in one of the earliest papers on the subject [6]; here instead, we derive it from the study of the periods near the engineering limit. It was conjectured already in [11] that the relation should have a natural interpretation in supergravity since $\log \Lambda$ is no longer an external parameter but becomes a VEV of a field in supergravity.

The relation should also follow from the microscopic calculations: Recall the fundamental relation in the path integral which states that

$$\langle \partial_\lambda L_0 \rangle = \partial_\lambda L_{\text{eff}} \tag{5.2}$$

where L_0 is the bare Lagrangian and L_{eff} is the low-energy effective Lagrangian including the quantum correction. λ denotes some coupling constant of the theory. In the case of supersymmetric theories one can likewise show

$$\langle \partial_\lambda W_0 \rangle = \partial_\lambda W_{\text{eff}}, \qquad \langle \partial_\lambda F_0 \rangle = \partial_\lambda F_{\text{eff}} \tag{5.3}$$

where W and F are the super and prepotential, respectively. Now in the $\mathcal{N} = 2$ theory $F_0 = \tau_0 \operatorname{tr} \phi^2$. Then the relation (5.1) follows because $\log \Lambda \propto \tau_0$ is the bare coupling constant and $u_2 = \langle \operatorname{tr} \phi^2 \rangle$. It can be seen more explicitly in the framework of multi-instanton calculation [21]. So the prepotential constructed from the SW curve should satisfy the relation.

Let us now turn to our derivation. Instead of the relation (5.1) itself, we shall show that its derivative with respect to the moduli satisfies

$$\frac{\partial^2 F_{\rm gauge}}{\partial u_j \partial \log\Lambda} = \frac{h}{\pi i}\,\delta_2^j. \tag{5.4}$$

Then (5.1) follows by integration. The integration constant is zero by virtue of the homogeneity of $F_{\rm gauge}$. The homogeneity can be used to rewrite l.h.s. of (5.4) and rewrite it as follows:

$$\frac{\partial^2 F_{\rm gauge}}{\partial u_j \partial \log\Lambda} = \frac{\partial}{\partial u_j}\left(2F_{\rm gauge} - \sum_i a^i \frac{\partial F_{\rm gauge}}{\partial a^i}\right) = \sum_i \frac{\partial a^i}{\partial u_j} a_i^D - \sum_i a^i \frac{\partial a_i^D}{\partial u_j}. \tag{5.5}$$

Now we use the relation (4.10) between the periods of rigid and local theory and obtain

$$\sum_i F_i \frac{\partial X^i}{\partial u_j} - \sum_i X^i \frac{\partial F_i}{\partial u_j} = \epsilon^{2/h} \frac{\partial^2 F_{\rm gauge}}{\partial u_j \partial \log\Lambda} + O(\epsilon^{3/h}) \tag{5.6}$$

One of the fundamental properties of the special geometry in supergravity is the transversality condition (2.18). We decompose the periods into two sets as $(X^i, F_i; X^a, F_a)$ where X^i, F_i are the periods which become those of the gauge theory of the rigid limit, and X^a, F_a are the extra periods in supergravity. We have

$$\sum_i F_i \frac{\partial X^i}{\partial u_j} - \sum_i X^i \frac{\partial F_i}{\partial u_j} = -\sum_a F_a \frac{\partial X^a}{\partial u_j} + \sum_a X^a \frac{\partial F_a}{\partial u_j}. \tag{5.7}$$

As shown in the previous section, X^a and F_a have at most $\log\epsilon$ singularity and the rest are analytic in $\epsilon^{i/h}u_i$. Furthermore, the logarithmic terms cancel in the r.h.s. of (5.7) because the l.h.s. is analytic in $\epsilon^{1/h}$. Therefore we have

$$\sum_i F_i \frac{\partial X^i}{\partial u_j} - \sum_i X^i \frac{\partial F_i}{\partial u_j} = -\sum_a F_a \frac{\partial X^a}{\partial u_j} + \sum_a X^a \frac{\partial F_a}{\partial u_j} = \text{const}\cdot\epsilon^{2/h}\,\delta_2^j + O(\epsilon^{3/h}). \tag{5.8}$$

Comparing with (5.6), we obtain (5.4) up to a constant factor.

Two comments are in order: first, the constant factor is non-trivial to determine in general but should be straightforward to fix in specific cases. It then fixes the proportionality factor between u_2 entering in the geometry and $\langle \mathrm{tr}\,\phi^2\rangle$. Second, the derivation above was so simple that it makes us suspicious why a similar analysis cannot be done in the field theory limit. Indeed, r.h.s. of (5.5) is a monodromy-invariant quantity of mass dimension $2-j$. Thus, it is a rational function of u_j's of dimension $2-j$ and it is forced to be δ_j^2 once one can argue it does not have poles. This is precisely the hard part because the special coordinates a^i and a_i^D are complicated functions of u_j's with a lot of cuts. In our derivation, we utilize the fact that the extra supergravity periods are analytic in $\epsilon^{j/h}u_j$, which does the job.

JHEP08(2007)068

6. Discussion

In this article, we have seen how the holomorphy inherent in $\mathcal{N}=2$ supersymmetry can be effectively used to study the effect of gravity upon the running of gauge theory. More specifically, we showed how the monodromy of the periods around the locus of the rigid limit translates to the hierarchical separation of the dynamical scale of gauge theory and the Planck scale. We have argued that, as compared to the naive relation

$$\Lambda_{\text{gauge}} \approx e^{-4\pi^2/hg^2}\ M_{\text{pl}} \tag{6.1}$$

there is generically an extra factor of the gauge coupling constant g in the right hand side,

$$\Lambda_{\text{gauge}} \approx e^{-4\pi^2/hg^2}\ \cdot gM_{\text{pl}} \tag{6.2}$$

supporting the weak gravity conjecture. We have also seen how the scaling violation of the prepotential of the gauge theory, (5.1), can be naturally understood from the embedding into supergravity.

The result presented here is only a small step in utilizing the holomorphy to understand the dynamics of the coupled $\mathcal{N}=2$ supergravity-gauge systems. We believe many more properties can be learned in a similar manner. It would also be interesting to make a comparison with the result in [22] where the authors calculated the one-loop effect of gravity to the beta function of the gauge theory. It was argued in [23] that the beta function in [22] alone leads to the weak gravity conjecture. We will have to supersymmetrize the result of [22] to carry out the comparison to our case.

It will be very important to see if it is possible to extend our results to the realm of $\mathcal{N}=1$ supersymmetric theories. In the case when $\mathcal{N}=1$ theories are obtained from those of $\mathcal{N}=2$ by introducing fluxes, branes etc. many of the structures of the latter survive. Hopefully we will have enough control over mass scales of these theories to derive the characterization of consistent $\mathcal{N}=1$ field theories coupled to gravity.

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